THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MMAT5540 Advanced Geometry 2016-2017
Supplementary Exercise 4

1. Find the equation of circle passing through the points $5+i$, $-3+5i$ and $4-2i$.

2. Let $f(z)=\dfrac{1}{\bar{z}}$ and let w_1, w_2, w_3, w_4 be four distinct complex numbers.

   Show that $[f(w_1), f(w_2), f(w_3), f(w_4)] = \overline{[w_1, w_2, w_3, w_4]}$ and so $f(z)$ maps a circle or a line to a circle or a line.

3. Let $\mathbb{D}=\{z\in\mathbb{C} : |z|<1\}$ and $\mathrm{Aut}(\mathbb{D})=\{f(z)=\lambda\dfrac{z-a}{\bar{a}z-1} \,:\, a, \lambda\in\mathbb{C}, |a|<1, |\lambda|=1\}$.

   Let $f(z)\in\mathrm{Aut}(\mathbb{D})$. Show that

   (a) if $|z|<1$, then $|f(z)|<1$ and if $|z|=1$, then $|f(z)|=1$;

   (b) if l is a straight line passing through 0, then the image of l under f is also a straight line passing through 0.

   (c) if γ is a circle perpendicular to the unit circle $\Gamma=\{z\in\mathbb{C} : |z|=1\}$, then the image of γ under f is also a circle perpendicular to Γ.

4. Let $z=x+iy\in\mathbb{C}$ where $x,y\in\mathbb{R}$ and let $f(z)=\dfrac{z-\frac{i}{2}}{(-\frac{i}{2})z-1}=\dfrac{2z-i}{-iz-1}$.

   Find the images of the straight line $x=y$ and the circle $x^2+y^2=1/16$ under $f(z)$.

   (Remark: Do their images perpendicular to each other at every intersection point?)

Lecturer's comment:

1. Let $z=x+iy\in\mathbb{C}$, where $x,y\in\mathbb{R}$. Ssuppose z lies on the circle passing through the points $5+i$, $-3+5i$ and $4-2i$.

   Then, we have $[5+i,-3+5i,4-2i,z]\in\mathbb{R}$, i.e. $\mathrm{Im}([5+i,-3+5i,4-2i,z])=0$.

$$\begin{aligned}
[5+i,-3+5i,4-2i,z] &= \left(\frac{(x+yi)-(-3+5i)}{(5+i)-(-3+5i)}\right) \Big/ \left(\frac{(x+yi)-(4-2i)}{(5+i)-(4-2i)}\right) \\
&= \left(\frac{(x+3)+(y-5)i}{8-4i}\right)\cdot\left(\frac{1+3i}{(x-4)+(y+2)i}\right) \\
&= \frac{(x-3y+18)+(3x+y+4)i}{(8x+4y-24)+(-4x+8y+32)i} \\
\mathrm{Im}([5+i,-3+5i,4-2i,z]) &= \frac{-(x-3y+18)(-4x+8y+32)+(3x+y+4)(8x+4y-24)}{(8x+4y-24)^2+(-4x+8y+32)^2}
\end{aligned}$$

   Therefore, $\mathrm{Im}([5+i,-3+5i,4-2i,z])=0$ implies $-(x-3y+18)(-4x+8y+32)+(3x+y+4)(8x+4y-24)=28(x^2+y^2-2y-24)=0$.

   The required circle is $x^2+y^2-2y-24=0$

2. We have

$$\begin{aligned}
[f(w_1), f(w_2), f(w_3), f(w_4)] &= \left(\frac{f(w_4)-f(w_2)}{f(w_1)-f(w_2)}\right) \Big/ \left(\frac{f(w_4)-f(w_3)}{f(w_1)-f(w_3)}\right) \\
&= \left(\frac{\frac{1}{\overline{w_4}}-\frac{1}{\overline{w_2}}}{\frac{1}{\overline{w_1}}-\frac{1}{\overline{w_2}}}\right) \Big/ \left(\frac{\frac{1}{\overline{w_4}}-\frac{1}{\overline{w_3}}}{\frac{1}{\overline{w_1}}-\frac{1}{\overline{w_3}}}\right) \\
&= \left(\frac{\frac{\overline{w_2}-\overline{w_4}}{\overline{w_2}\,\overline{w_4}}}{\frac{\overline{w_2}-\overline{w_1}}{\overline{w_1}\,\overline{w_2}}}\right) \Big/ \left(\frac{\frac{\overline{w_3}-\overline{w_4}}{\overline{w_3}\,\overline{w_4}}}{\frac{\overline{w_3}-\overline{w_1}}{\overline{w_1}\,\overline{w_3}}}\right) \\
&= \left(\frac{\overline{w_2}-\overline{w_4}}{\overline{w_2}-\overline{w_1}}\right) \Big/ \left(\frac{\overline{w_3}-\overline{w_4}}{\overline{w_3}-\overline{w_1}}\right) \\
&= \overline{[w_1, w_2, w_3, w_4]}
\end{aligned}$$

The equation of a circle or a line can be given by $\mathrm{Im}([z_1, z_2, z_3, z]) = 0$, where z_1, z_2, z_3 are three distinct points on that circle or line.

Let $w = f(z)$. By the above, we have

$\mathrm{Im}([f(z_1), f(z_2), f(z_3), w]) = \mathrm{Im}([f(z_1), f(z_2), f(z_3), f(z)]) = \mathrm{Im}(\overline{[z_1, z_2, z_3, z]}) = 0.$

Therefore, the image is still a circle or a line passing through the points $f(z_1), f(z_2), f(z_3)$.

3. Let $w = f(z) = \lambda \dfrac{z-a}{\overline{a}z-1}$, where $a, \lambda \in \mathbb{C}$, $|a| < 1$ and $|\lambda| = 1$.

   (a) If $|z| = 1$, then

$$\begin{aligned}
|w|^2 &= w\overline{w} \\
&= \left(\lambda\frac{z-a}{\overline{a}z-1}\right)\left(\overline{\lambda}\frac{\overline{z}-\overline{a}}{a\overline{z}-1}\right) \\
&= \lambda\overline{\lambda}\frac{|z|^2 - a\overline{z} - \overline{a}z + |a|^2}{|a|^2|z|^2 - a\overline{z} - \overline{a}z + 1} \\
&= 1
\end{aligned}$$

   The last equality follows from the fact that $|\lambda| = 1$ and the assumption that $|z| = 1$. Then, $|w|^2 = 1$ implies $|w| = 1$.

   (b) If $|z| < 1$, then

$$\begin{aligned}
w &= \lambda\frac{z-a}{\overline{a}z-1} \\
z &= \frac{w-\lambda}{\overline{a}w-\lambda} \\
1 > |z| &= \left|\frac{w-\lambda a}{\overline{a}w-\lambda}\right| \\
|\overline{a}w-\lambda| &> |w-\lambda a| \\
|\overline{a}w-\lambda|^2 &> |w-\lambda a|^2 \\
(\overline{a}w-\lambda)(a\overline{w}-\overline{\lambda}) &> (w-\lambda a)(\overline{w}-\overline{\lambda a}) \\
|a|^2|w^2| - \lambda a\overline{w} - \overline{\lambda a}w + 1 &> |w^2| - \lambda a\overline{w} - \overline{\lambda a}w + |\lambda|^2|w|^2 \\
1 &> |w|^2 \\
|w| &< 1
\end{aligned}$$

(c) If γ is a circle perpendicular to the unit circle Γ, then the image of γ under inversion is itself. In particular, we can choose a pair of points z_0 and $\dfrac{1}{\overline{z_0}}$ such that both of them are lying on γ.

Then, $f(z_0) = \lambda\dfrac{z_0 - a}{\overline{a}z_0 - 1}$ and $f(\dfrac{1}{\overline{z_0}}) = \lambda\dfrac{(\frac{1}{\overline{z_0}}) - a}{\overline{a}(\frac{1}{\overline{z_0}}) - 1} = \lambda\dfrac{1 - a\overline{z_0}}{\overline{a} - \overline{z_0}}$.

We also note that $\dfrac{1}{\overline{f(z_0)}} = \dfrac{1}{\overline{\lambda}} \cdot \dfrac{a\overline{z_0} - 1}{\overline{z_0} - \overline{a}} = (-\dfrac{1}{\overline{\lambda}}) \cdot \dfrac{1 - a\overline{z_0}}{\overline{a} - \overline{z_0}} = \lambda\dfrac{1 - a\overline{z_0}}{\overline{a} - \overline{z_0}} = f(\dfrac{1}{\overline{z_0}})$.

Therefore, the image of γ under $f(z)$ is a circle that contains $f(z_0)$ and $\dfrac{1}{\overline{f(z_0)}} = f(\dfrac{1}{\overline{z_0}})$, which shows that the image of γ under $f(z)$ is a circle is again perpendicular to Γ.

4. Let $w = u + iv = f(z)$, where $u, v \in \mathbb{R}$. Then,

$$
\begin{aligned}
w &= \frac{2z - i}{-iz - 1} \\
z &= \frac{-w + i}{2 + iw} \\
x + iy &= \frac{-u + (-v + 1)i}{(2 - v) + iu} \\
&= \frac{-u + (u^2 + v^2 - 3v + 2)i}{(2 - v)^2 + u^2}
\end{aligned}
$$

Therefore, $x = \dfrac{-u}{(2 - v)^2 + u^2}$ and $y = \dfrac{u^2 + v^2 - 3v + 2}{(2 - v)^2 + u^2}$.

If $x = y$, then we have $u^2 + v^2 + u - 3v + 2 = 0$, which is the image of $x = y$ under $f(z)$.

If $x^2 + y^2 = 1/16$, then $15u^2 + 15v^2 - 28v + 12 = 0$, which is the image of $x^2 + y^2 = 1/16$ under $f(z)$.

(Remark: Let $\mathcal{C}_1$ and $\mathcal{C}_\in$ be two circles on a plane with radius r_1 and r_2 respectively and let d be the distance between two centers. Two circles are perpendicular if and only if $r_1^2 + r_2^2 = d^2$. You may check the above two circles are perpendicular to each other.)